New family of models for incompressible quantum liquids in $d \geq 2$

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Through Haldane's construction, the fractional quantum Hall states on a two-sphere was shown to be the ground states of *one-dimensional* SU(2) spin Hamiltonians. In this Letter we generalize this construction to obtain a new class of SU(N) spin Hamiltonians. These Hamiltonians describes center-of-mass-position conserving pair hopping fermions in space dimension $d \geq 2$.

Gapped quantum many-body systems are stable states of matter because they are robust against weak perturbations. For fermions, the most common gapped system is the "band insulator" where an energy gap in the dispersion relation separates filled and empty single-particle states. Here the insulating behavior is caused by the Pauli exclusion principle. We refer to a system as a "many-body insulator" if its energy gap is caused by many-body interaction rather than one-particle dispersion relation. It is widely believed that such a many-body gap can exist when the occupation number (i.e.,the averaged number of particle per spin per unit cell) is a fraction. The Mott insulator, where the energy gap is due to the inter-particle repulsion, is an example of many-body insulator.

Contrary to the common belief, it is very difficult to find true many-body insulators. In most cases an energy gap at fractional occupation number is accompanied by the breaking of translation symmetry. After symmetry breaking, the unit cell is enlarged such that the new occupation number is an integer. In a recent work Oshikawa argued that the existence of energy gap at fractional occupation requires the ground state to be degenerate[1]. It happens that under usual circumstances such degeneracy is caused by translation symmetry breaking.

The fractional quantum Hall state is a quantum liquid (hence no breaking of translation symmetry) with an energy gap. On the surface it is not clear what does it have to do with the many-body insulator discussed above. In Ref.[2–4] it is shown that, when placed on a torus, both abelian and non-abelian quantum Hall liquids can be mapped to many-body insulators on a *one dimensional* ring of lattice sites. When both dimensions of the torus are much bigger than the magnetic length, the energy gap is generated by a long range center-of-mass-position conserving hopping, rather than density-density interaction.

Partly motivated by Anderson's proposal of spin liquid[5], the question of whether a many-body insulator can exist without symmetry breaking in spatial dimension $d \geq 2$ has attracted a lot of interests. In this Letter we give this question an affirmative answer by explicitly constructing a new class of solvable lattice models that exhibit incompressible quantum liquid ground states. Since our construction is a generalization of Haldane's work on the pseudopotential Hamiltonian for the fractional quantum Hall effect[6], we shall begin by briefly review it.

If a magnetic monopole of strength $2S$ (S is a multiple of 1/2) is placed at the center of a two-sphere, the kinetic energy spectrum of a particle confined to move on the sphere is given by $E_k \propto (S+k)(S+k+1)$, where k is an non-negative integer. The kth "Landau level" is $2(S+k)+1$-fold degenerate. The degeneracy of the lowest Landau level, $k=0$, is exactly the dimension of a spin-S SU(2) multiplet. Thus the Hilbert space of N spin polarized electrons in the lowest Landau level is the same as the exchange-antisymmetric sub-Hilbert space of N such SU(2) spins. Haldane's pseudopotential Hamiltonian (which has the spherical version of Laughlin's $\nu = 1/m$ wavefunction as the ground state) is given by

$$H = \frac{1}{2}\sum_{i\neq j}\sum_{q=1,odd}^{m-2}\kappa_q\, P_{ij}^{2S-q}. \tag{1}$$

Here $i,j=1,..,N$ are the spin labels, P_{ij}^{2S-q} projects the product states of spin i and j onto the total spin $2S-q$ multiplet, and $\kappa_q > 0$ are parameters (as a result Eq. (1) is positive-definite). In Eq.(1) the q-sum is restricted to odd integers because the restriction of the Hilbert space to the total antisymmetric subspace. It can be shown that when $N-1 = 2S/m \equiv p$ the Hamiltonian in Eq.(1) has an unique ground state described by the following spin coherent-state wavefunction(the spherical version of Laughlin's wavefunction)

$$\Psi = \begin{vmatrix} u_1^p & u_1^{p-1}v_1 & . & . & v_1^p \\ u_2^p & u_2^{p-1}v_2 & . & . & v_2^p \\ . & . & . & . & . \\ . & . & . & . & . \\ u_N^p & u_N^{p-1}v_N & . & . & v_N^p \end{vmatrix}^m \tag{2}$$

Upon expanding the determinant, Eq.(2) can be written as a linear combination of $\prod_{j=1}^N \sqrt{(mp)!/n_j!k_j!}\,u_j^{n_j}v_j^{k_j}$ where $n_j+k_j = mp$. Each term in this linear combination is a direct product of N, spin-S, states. An important property of the wavefunction in Eq.(2) is that the highest total spin for any pair is $2S-m$. Consequently Eq.(2) is a zero energy state of Eq. (1). Moreover, if we

view the $2S+1$ different S_z states as the $2S+1$ local orbitals of a one-dimensional lattice, and write $P_{ij}^{2S-q} = \sum_{l=-(2S-q)}^{2S-q} \sum_{m_2=-S}^{S} \sum_{m_1=-S}^{S} C_{S,m_1,S,l-m_1}^{2S-q,l} C_{S,m_2,S,l-m_2}^{2S-q,l} |m_2, l-m_2 >< m_1, l-m_1|$, Eq.(1) becomes a center-of-mass conserving pair hopping Hamiltonian.[2, 3] In the above $C_{S,m_1,S,l-m_1}^{2S-q,l}$ is the SU(2) Clebsch-Gordon coefficient. The role of center-of-mass position conservation in producing true many-body insulators was discussed in Ref.[2, 3].

In the following we generalize Haldane's construction to SU(3) spins. The reason for doing so is SU(3), a rank two Lie group, has multiplets isomorphic to *two dimensional* lattices. Hence it allows us the possibility of constructing Hamiltonians for many-body insulator in $d=2$. The irreducible representations of SU(3) are labelled by two integers (p,q). In the following we shall focus on the the multiplets $(k,0)$. The reason is because these multiplets are the only ones whose weight space is an array of *non-degenerate*, i.e., non-duplicated, points (see Fig.(1)). In the following we consider SU(3) spins each in the $(mp,0)$ representation. The dimension of the $(mp,0)$ representation is $d(mp)=(mp+1)(mp+2)/2$. For reason that shall become clear later, we shall choose the number of spins so that $N=d(p)=(p+1)(p+2)/2$. Under such condition the filling factor, $f=d(p)/d(mp)$, is $1/m^2$ in the thermodynamic ($p\rightarrow\infty$) limit. As earlier, we will constrain the N-spin Hilbert space to be exchange-antisymmetric to mimic the fermion statistics.

The spin Hamiltonian we construct is a generalization of Eq. (1), and is given by

$$H = \frac{1}{2}\sum_{i\neq j}\sum_{q=1,\ \mathrm{odd}}^{q\leq m-2} \kappa_q\ P_{ij}^{(2mp-2q,q)}. \tag{3}$$

Here the operator $P_{ij}^{(2mp-2q,q)}$ operates on the direct product states of two spins i and j, and projects them onto the $(2mp-2q,q)$ multiplet, and $\kappa_q>0$. For simplicity in the rest of the Letter we shall set $m=3$ and $p=$ odd integer. In this case Eq.(3) becomes

$$H = \frac{\kappa_1}{2}\sum_{i\neq j} P_{ij}^{(6p-2,1)}. \tag{4}$$

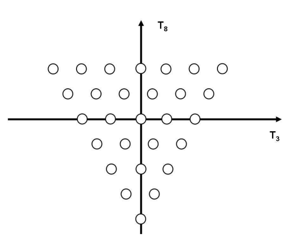

FIG. 1: The weight space of $(6,0)$.

We shall spend much of the rest of the Letter to prove that Eq.(4) has a unique singlet ground state described by the following SU(3) coherent-state wavefunction

$$\Psi_3 = \begin{vmatrix} u_1^p & u_1^{p-1}v_1 & . & . & w_1^p \\ u_2^p & u_2^{p-1}v_2 & . & . & w_2^p \\ . & . & . & . & . \\ . & . & . & . & . \\ u_N^p & u_N^{p-1}v_N & . & . & w_N^p \end{vmatrix}^3 \tag{5}$$

First we prove Eq.(5) is a ground state. Let us focus on the dependence of Eq. (5) on the variables of any chosen pair of spin i and j. For each of the Slater determinant in Eq. (5) the highest total SU(3) weight of these two spins is $(2p-2,1)$ because of antisymmetry. As a result when we multiply three determinant together the highest total SU(3) weight for spin i and j is $(6p-6,3)$. Consequently Eq.(5) is an zero-energy eigenstate of the positive-definite Hamiltonian in Eq.(4). Thus we have found a ground state.

As to the uniqueness let us start with the simplest case of $p=1$ where the single spin Hilbert space is $d(3)=10$ dimensional and there are $d(1)=3$ spins. The most general many-body wave function is given by

$$\chi = \sum_{\{\alpha_j,\beta_j\gamma_j=1\}}^{3} C(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3) \times\phi_1^{\alpha_1}\phi_1^{\alpha_2}\phi_1^{\alpha_3}\phi_2^{\beta_1}\phi_2^{\beta_2}\phi_2^{\beta_3}\phi_3^{\gamma_1}\phi_3^{\gamma_2}\phi_3^{\gamma_3}, \tag{6}$$

where $\phi_j^{1,2,3} = u_j, v_j, w_j$. The requirement that this wavefunction lies in the direct product of $(3,0)\otimes(3,0)\otimes(3,0)$ demands the coefficient C to be invariant when the three indices of any chosen particle are permuted. In addition, the antisymmetry constraint restricts C to change sign upon the exchange of particle labels. Next, let us pick any pair of spins i and j and examine the dependence of Eq. (6) on their variables. The possible total SU(3) weights of these two spins that are consistent with antisymmetry are given by $(3,0)\otimes(3,0)=(4,1)\oplus(0,3)$. In order for the wavefunction to be annihilated by Eq. (4), it must lie entirely in $(0,3)$. Since there are only three particles it is not hard to show that there is a unique C so that the above condition holds for all pair (i,j):

$$C(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3) \propto \epsilon_{\alpha_1\beta_1\gamma_1}\epsilon_{\alpha_2\beta_2\gamma_2}\epsilon_{\alpha_3\beta_3\gamma_3}.$$

Substitute the above equation into Eq. (6) gives $\chi\sim\Psi_3$.

Now, we consider $p=3$, where the single-particle Hilbert space is 55 dimensional and and there are $d(3)=10$ particles. Analogous to Eq. (6) the most general 10-particle wavefunction is given by

$$\chi = \sum_{\{\alpha_{jn}=1\}}^{3} C(\{\alpha_{jn}\})\prod_{j=1}^{10}\prod_{n=1}^{9}\phi_j^{\alpha_{jn}}. \tag{7}$$

Here j is the particle label, $n=1,...,9$ labels the nine fundamental SU(3) spinors $(1,0)$ that make up $(9,0)$.

The symmetry properties of C are the same as before. In this case the total SU(3) weight of any two spins that are consistent with antisymmetry are given $(9,0)\otimes(9,0)=(16,1)\oplus(12,3)\oplus(8,5)\oplus(4,7)\oplus(0,9)$. The condition of being annihilated by the Hamiltonian requires the ground state wavefunction to lie entirely in $(12,3)\oplus(8,5)\oplus(4,7)\oplus(0,9)$, i.e.,

$$(P_{ij}^{(12,3)}+P_{ij}^{(8,5)}+P_{ij}^{(4,7)}+P_{ij}^{(0,9)})\chi=\chi,\quad \forall\ (i,j). \quad (8)$$

Eq. (8) implies that among the 9 indices spin i and j each possesses, there must be at least 3 pairs (a pair contains one index from each particle) such that $C\to -C$ upon exchanging the indices within each pair. In addition to be consistent with exchange antisymmetry, the number of such pairs also must be odd. Since C is invariant when the nine indices of any particle are permuted, we can perform permutations so that in each triplet $(\alpha_{i1},\alpha_{i2},\alpha_{i3})$, $(\alpha_{i4},\alpha_{i5},\alpha_{i6})$,$(\alpha_{i7},\alpha_{i8},\alpha_{i9})$ of particle i and $(\alpha_{j1},\alpha_{j2},\alpha_{j3})$, $(\alpha_{j4},\alpha_{j5},\alpha_{j6})$,$(\alpha_{j7},\alpha_{j8},\alpha_{j9})$ of particle j there is an *odd number* of antisymmetric indices. Under such circumstance C changes sign upon *independent* exchanges of the triplets, i.e.,

$$\begin{aligned} C\to -C\quad \text{upon}\quad &(\alpha_{i1},\alpha_{i2},\alpha_{i3})\leftrightarrow(\alpha_{j1},\alpha_{j2},\alpha_{j3})\\ &(\alpha_{i4},\alpha_{i5},\alpha_{i6})\leftrightarrow(\alpha_{j4},\alpha_{j5},\alpha_{j6})\\ &(\alpha_{i7},\alpha_{i8},\alpha_{i9})\leftrightarrow(\alpha_{j7},\alpha_{j8},\alpha_{j9}). \quad (9)\end{aligned}$$

In Eq. (9) $(...)\leftrightarrow(...)$ denotes the exchange of whole group of indices. If Eq. (9) can be made true *simultaneously* for all pairs i and j then Ψ_3 is the unique solution of Eq. (8). This is proven as follows.

Let us focus on the dependence of C on the first index triplet of all particle. For each triplet of indices, say $(\alpha_{i1},\alpha_{i2},\alpha_{i3})$, there are $(3+2)!/(3!2!)=d(3)=10$ inequivalent combinations. We can interpret each combination as a single-particle quantum state and C as the wavefunction for 10 particles to occupy these states. The first line of Eq. (9) allows us to interpret C as the wavefunction for fermions. For $N=d(3)=10$, i.e., when the fermion number is the same as the number of single-particle state, there is an unique wavefunction satisfying the antisymmetric requirement, namely,

$$C(\{\alpha_{i1},\alpha_{i2},\alpha_{i3}\}...)\sim\epsilon\{(\alpha_{i1}\alpha_{i2}\alpha_{i3})\} \quad (10)$$

where $\epsilon\{...\}$ is the rank 10 total antisymmetric tensor with respect to the exchange of index-triplets. Similar argument can be made to $\{(\alpha_{i4},\alpha_{i5},\alpha_{i6})\}$ and $\{(\alpha_{i7},\alpha_{i8},\alpha_{i9})\}$, and lead to

$$\begin{aligned} &C(\{\alpha_{i1},...,\alpha_{i9}\})\sim\epsilon\{(\alpha_{i1}\alpha_{i2}\alpha_{i3})\}\epsilon\{(\alpha_{i4}\alpha_{i5}\alpha_{i6})\}\\ &\times\ \epsilon\{(\alpha_{i7}\alpha_{i8}\alpha_{i9})\}. \quad (11)\end{aligned}$$

Substitute Eq. (11) into Eq. (7) we obtain $\chi\sim\Psi_3$.

Now, we shall prove that Eq.(9) can indeed be made true for all pairs i and j simultaneously. Let us assume that there exists a ground state solution whose C does not satisfy Eq.(9) for pair (k,l). This means there must be at least one triplet exchange, let say $\{\alpha_{k1},\alpha_{k2},\alpha_{k3}\}\leftrightarrow\{\alpha_{l1},\alpha_{l2},\alpha_{l3}\}$, for which C does not transform according to Eq. (9). However, since the wavefunction still has to satisfy Eq. (8) for (k,l), we should be able to write $C=C_3+C_5+C_7+C_9$, where C_q is the component of C that is odd with respect to exchange of *exactly* q pair of indices between particle k and l and even with respect to the exchange of the rest. Now let us consider the effect of $\{\alpha_{k1},\alpha_{k2},\alpha_{k3}\}\leftrightarrow\{\alpha_{l1},\alpha_{l2},\alpha_{l3}\}$ on C. Under such operation C_q can either change sign or stay invariant depending on whether an odd or even number (out of q) antisymmetric indices are contained in the specified triplets. In other words upon $\{\alpha_{k1},\alpha_{k2},\alpha_{k3}\}\leftrightarrow\{\alpha_{l1},\alpha_{l2},\alpha_{l3}\}$ we have

$$C\to\eta_3C_3+\eta_5C_5+\eta_7C_7+\eta_9C_9, \quad (12)$$

where $\eta_q=\pm1$. Since Eq.(9) is not satisfied, $\eta_{3,5,7,9}$ must not simultaneously be -1. Now consider a new C

$$C'\equiv\frac{1}{2}\Big[C-\eta_3C_3-\eta_5C_5-\eta_7C_7-\eta_9C_9\Big]. \quad (13)$$

It is obvious that upon $\{\alpha_{k1},\alpha_{k2},\alpha_{k3}\}\leftrightarrow\{\alpha_{l1},\alpha_{l2},\alpha_{l3}\}$ $C'\to -C'$. Moreover by construction C' only contains those C_q whose $\eta_q=-1$. Now use C' as the starting C and repeat the above operation until we reach a final C' for which Eq.(9) holds for all triplet exchanges and for all (i,j). Since at each stage of obtaining C' certain C_q are projected out, there must be missing q components in the final C. However we have already proven that any C that satisfy Eq.(9) for all (i,j) pair must satisfy Eq. (11). However Eq. (11) contains all four q components for all pair (i,j). Consequently we have reached a contradiction. Therefore it must be possible to make Eq.(9) hold true for all pairs (i,j) for any ground state solution satisfying Eq. (8).

Although we have chosen $p=3$ and $m=3$ in the above discussion, it should be clear that our proof can be generalized to any odd p , any m. Thus we have proven that Eq.(5) is the unique ground state of Eq.(4).

It is straightforward to prove that Eq.(3) is a center-of-mass position conserving pair hopping model, i.e.,

$$\begin{aligned} H\ &=\kappa\sum_{j,L,L_3}\sum_{l,l_3}\sum_{k,k_3}F_{l,l_3}^{j,L,L_3}F_{k,k_3}^{j,L,L_3}\\ &c^\dagger_{(l,l_3)}c^\dagger_{(j-l+\frac{1}{2},L_3-l_3)}c_{(j-k+\frac{1}{2},L_3-k_3)}c_{(k,k_3)}. \quad (14)\end{aligned}$$

The central steps are 1) viewing the weight space of $(mp,0)$ as a triangular lattice, and 2) decomposing the two spin states in Eq. (3) as linear combination of products of single spin state. Due to space limitation, the result will be published elsewhere. Using the explicit expression for the F's in Eq. (14) (lengthy hence is omitted here) we can estimate the hopping range. In Fig.(2) we

plot the hopping range versus p for $p = 30 \to p = 600$. While the linear dimension of the lattice scales as p, the hopping range scales as $p^{\frac{1}{2}}$, hence the hopping is long-ranged as the SU(2) case.

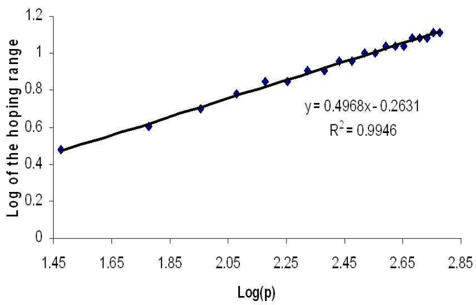

FIG. 2: A log-log plot of the hopping range and p from 30 to 600. The straight line is the best fit. The vertical axis is the log of the hopping range and the horizontal one is $\log p$. The hopping range scales as $p^{\frac{1}{2}}$

Finally, we demonstrate the presence of an excitation gap within the single mode approximation (SMA)[7]. Analogous to the SU(2) case it is possible to view the $(k, 0)$ SU(3) multiplet as the "lowest Landau level"(LLL) of a particle running on CP^2 under the action of a U(1) background magnetic field[8]. If we parameterize the fundamental SU(3) spinor as $(1, z_1, z_2)/\sqrt{1 + |z_1|^2 + |z_2|^2}$, where $z_i = x_i + iy_i$, and take the flat-space limit (i.e., restrict $|z_{1,2}| << 1$) the single-particle orbitals in the LLL become

$$\Phi_{l_1,l_2}(z_1, z_2) = \frac{1}{\sqrt{4\pi^2 2^{l_1} 2^{l_2} l_1! l_2!}} z_1^{l_1} z_2^{l_2} e^{-(|z_1|^2+|z_2|^2)/4},$$

where $l_{1,2}$ are non-negative integers. We recognize that the above result is the product of two LLL wavefunctions in two space dimensions. Thus in the flat-space limit, the LLL in CP^2 becomes the direct product of the LLLs in two quantum Hall planes. This reduction allows us to perform the SMA calculation pretty much in parallel to that for the ordinary quantum Hall effect.[7, 9] in the following we summarize the results. Within SMA the excitation energy is given by

$$\Delta(\mathbf{k}) = f(\mathbf{k})/s(\mathbf{k}). \tag{15}$$

Here $\mathbf{k} = (k_1, k_2)$ where $k_{1,2}$ are the complex wave vectors associated with the two quantum Hall planes, and $f(\mathbf{k})$ and $s(\mathbf{k})$ are given by

$$\begin{aligned} f(\mathbf{k}) &= (1/N) < \Psi_m|[\rho^\dagger_{\mathbf{k}}, [V, \rho_{\mathbf{k}}]]|\Psi_m > \\ s(\mathbf{k}) &= (1/N) < \Psi_m|\rho^\dagger_{\mathbf{k}}\rho_{\mathbf{k}}|\Psi_m > . \end{aligned} \tag{16}$$

In the above equation $\rho_{\mathbf{k}}$ and V are the the density operator and the inter-particle potential projected onto the LLL. Straightforward calculation gives,

$$\begin{aligned} f(\mathbf{k}) = \frac{1}{2}\sum_{\mathbf{q}} v(|\mathbf{q}|)(e^{\frac{\bar{\mathbf{q}}\mathbf{k}}{2}} - e^{\frac{\bar{\mathbf{k}}\mathbf{q}}{2}})[s(\mathbf{q})e^{-\frac{|\mathbf{k}|^2}{2}}(e^{-\frac{\bar{\mathbf{k}}\mathbf{q}}{2}} - e^{-\frac{\bar{\mathbf{q}}\mathbf{k}}{2}}) \\ + s(\mathbf{k}+\mathbf{q})(e^{\frac{\bar{\mathbf{k}}\mathbf{q}}{2}} - e^{\frac{\bar{\mathbf{q}}\mathbf{k}}{2}})], \end{aligned} \tag{17}$$

where $v(|\mathbf{q}|)$ is the Fourier transformation of the potential, which is required to be positive indicating the repulsive interaction to ensure the excitation energy to be positive. On the other hand, $s(\mathbf{k})$ can be related to the radial distribution function $g(\vec{r})$ by

$$s(\mathbf{k}) = e^{-\frac{|\mathbf{k}|^2}{2}} + \rho \int d^4 r e^{-i\vec{k}\cdot\vec{r}}[g(\vec{r}) - 1] + \rho(2\pi)^4 \delta^4(\vec{k}) \tag{18}$$

where ρ is the average density and $\vec{k}$=(Re(k_1), Im(k_1), Re(k_2), Im(k_2)). After some algebra, for small $|\mathbf{k}|$ it can be shown that $\Delta(\mathbf{k}) = (a|k_1|^4 + b|k_1|^2|k_2|^2 + a|k_2|^4)/(c|k_1|^4 + d|k_1|^2|k_2|^2 + c|k_2|^4)$, which remains finite as $\mathbf{k}$ approaches to zero in any direction.

In summary, we have constructed a two dimensional center-of-mass conserving pair hopping model which exhibit incompressible quantum liquid ground state. Although throughout the Letter we have focused on SU(3) whose weight space is two dimensional, our construction can easily be generalized to SU(N) giving rise to models for incompressible quantum liquid in higher dimensions. Particularly, the SU(4) model can be applied to the four-dimensional quantum Hall effect proposed by Zhang and Hu [10–12].

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